Forces Associated with Nonlinear Nonholonomic Constraint Equations

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Abstract

A concise method has been formulated for identifying a set of forces needed to constrain the behavior of a mechanical system, modeled as a set of particles and rigid bodies, when it is subject to motion constraints described by nonholonomic equations that are inherently nonlinear in velocity. An expression in vector form is obtained for each force; a direction is determined, together with the point of application. This result is a consequence of expressing constraint equations with dot products of vectors rather than entirely in terms of scalars and matrices in the usual way. The constraint forces in vector form are used together with two new analytical approaches for deriving equations governing motion of a system subject to such constraints. If constraint forces are of interest they can be brought into evidence in explicit dynamical equations by employing the well-known nonholonomic partial velocities associated with Kane's method; if they are not of interest, equations can be formed instead with the aid of vectors introduced here as nonholonomic partial accelerations.

1 Introduction

Motion constraints imposed on a mechanical system are described with nonholonomic (non-integrable) constraint equations, whereas configuration constraints are expressed with holonomic constraint equations. Two examples of motion constraints with which the reader may be familiar are the condition of rolling, which is the absence of slipping, and the restriction on velocity imposed by a sharp-edged blade. These constraints are sometimes described with

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equations written in the matrix form $\alpha u + \beta = 0$, where u is a column matrix of motion variables $u_1, \ldots, u_n$. Motion variables, also referred to as generalized speeds, are in general linear combinations of the time derivatives of generalized coordinates, $\dot{q}_1, \ldots, \dot{q}_n$. The distinguishing feature of such equations is that they are linear in the motion variables. However, one may consider motion constraints that must be described by relationships that are inherently nonlinear in the motion variables, having the form $f(q_1, \ldots, q_n, u_1, \ldots, u_n, t) = 0$. In Ref. [1] Bajodah et al. review some of the literature dealing with nonlinear nonholonomic constraint equations and consider it important to study them because they can arise in connection with servo-constraints or program constraints when a control system enters the picture. As explained in Refs. [2] and [3], such constraints are enforced by application of control forces as opposed to the forces present when bodies and particles come into contact with one another, as is the case with classical, passive constraints.

It is important to make a distinction between nonholonomic constraint equations that are inherently nonlinear in the motion variables, and those whose nonlinearity is contrived. In this paper the differentiation between the two types of equations is kept firmly in mind, whereas in the existing literature the distinction is made infrequently, if at all.

Golubev states in Ref. [4] that, as of yet, there is no example of a passive mechanical device that can compel a motion constraint described by an equation that is nonlinear in velocity. Roberson and Schwertassek note in Ref. [5] that all known motion constraints imposed on purely mechanical systems can be expressed with relationships that are linear in velocity variables. Unfortunately, the relationships in such situations are often artificially teased into nonlinear forms to create contrived examples used to illustrate a proposed procedure. For instance, a nonlinear equation is devised in Ref. [6] to describe the constraint imposed on a rolling disk. The well-known Appell-Hamel mechanism is studied and discussed, for example, in Refs. [1] and [7] – [12]. It is recognized in Refs. [1] and and [8] – [12] that the constraints imposed on this mechanical system can be expressed with linear relationships, but despite this the mechanism is used in Refs. [1], [11], and [12] to demonstrate the application of methods for dealing with nonlinear nonholonomic constraint equations. In Refs. [13] and [14], Zekovich offers several examples of systems in which the constraints can be described with nonlinear nonholonomic constraint equations. Each example involves planar motion of two particles connected by a massless rigid rod or by a massless prismatic joint. Sharp-edged blades are attached in various ways so as to cause the velocities of the particles in an inertial reference frame N to be parallel, equal in magnitude, or perpendicular. In what follows it is shown that the associated constraints can in fact be expressed with linear nonholonomic equations. However, when the particles are not physically connected and the constraints are dictated by means other than the blades, the relationships expressing such restrictions on the velocities are inherently nonlinear. Another case of planar motion of two particles with parallel velocities, which serves as an example in Refs. [8], [15], [16], and [17], is brought about with a device proposed by Benenti in Ref. [18]. Benenti's mechanism consists of six rigid rods, one revolute joint, two blades, and a number of prismatic joints. Eight prismatic joints appear to be indicated in the figure in Ref. [18]; however, it is believed that there must be two successive prismatic joints in each of the locations indicated if all rods are to be able to move relative to each other. In any case a purely mechanical system is involved and therefore, according to the observations in Refs. [4] and [5], the nonlinear nonholonomic equation used to describe the constraint must be regarded as contrived.

Whenever a motion constraint can be expressed entirely with linear nonholonomic constraint equations, it should be dealt with accordingly. Any number of approaches can be used to deal with the equations in their linear form; the exercise of cajoling such equations into a nonlinear appearance serves no useful purpose. The new approaches contained in this paper, and the examples of their application, are concerned strictly with inherently nonlinear nonholonomic constraint equations.

The literature contains several instances of motion constraints described by nonholonomic equations that are inherently nonlinear in velocity. Perhaps the simplest case, provided by Golubev in Ref. [4], involves a single particle P that is subject to a uniform gravitational field and moves in a vertical plane fixed in an inertial reference frame N. The magnitude of the velocity ${}^{N}\mathbf{v}^{P}$ of P in N is to remain constant. The particle thus constrained serves as a model of a robot manipulator tip used to spray-paint a wall or polish a surface. The same constraint is imposed in two variations of this problem. The first is studied by San in Ref. [19]

and by Bahar in Ref. [20]; P moves in three dimensions and a central (radial) gravitational field is produced by a second particle fixed in N. In the second variation, also studied in Ref. [20], P moves in three dimensions and gravitational force is replaced by an impressed force of arbitrary magnitude and direction. Additional complexity is present in two problems analyzed by Jankowski in Ref. [21]. A single particle P, moving in a vertical plane, is subject to a uniform gravitational field and air resistance; the magnitude of ${}^N\mathbf{v}^P$, or the magnitude of the acceleration ${}^N\mathbf{a}^P$ of P in N, must match a prescribed time history. The equation that describes the constraint in the latter problem is nonlinear in acceleration, rather than nonlinear in velocity. References [7], [15], [19], [22], and [23] include an example proposed by Appell in which P moves in three dimensions in the presence of a uniform gravitational field. The velocity ${}^N\mathbf{v}^P$ must satisfy the relationship ${v_3}^2 = a^2({v_1}^2 + {v_2}^2)$, where a is a constant and v_r are the dot products of ${}^N\mathbf{v}^P$ with a set of right-handed, mutually perpendicular unit vectors $\hat{\mathbf{n}}_r$ fixed in N $(r = 1, 2, 3)$. The special case of $a = 1$ is examined in Ref. [20], and also in Ref. [24] with gravitational force replaced by an impressed force of arbitrary magnitude and direction. (The title of Ref. [19] not withstanding, the foregoing two examples from that work in fact involve constraints that can be described by *first*-order nonlinear nonholonomic relationships, or second-order *linear* nonholonomic equations.)

Control of an inverted pendulum constitutes an example studied in Refs. [15] and [16]. A thin rigid rod moves in a vertical plane in the presence of a uniform gravitational field, with the lower end of the rod always in contact with a horizontal line. The system is referred to as Marle's servomechanism; as proposed in Ref. [7], an actuator controls the horizontal displacement of the rod's lower end according to some control law in order to keep the rod vertical. An earlier paper by Huston and Passerello (Ref. [25]) considers the more general case of balancing a pole whose lower end remains in contact with a horizontal plane, while the pole is otherwise free to move in the space above the horizontal plane.

The forthcoming developments in this paper are carried out for the most part in terms of vectors. These quantities are used also in expressing the main results, and discussing the contributions of the work. By *vector* we mean a basis-independent quantity having direction and magnitude, such as position, velocity, acceleration, or force, involved in the application of elementary principles of dynamics to study motion taking place in three-dimensional space. Other examples of a vector include partial velocities and partial angular velocities associated with advanced principles of dynamics. We do not mean a row or column matrix whose elements consist of three basis-dependent scalar measure numbers of a vector. Nor do we have in mind a matrix containing more than three scalar elements, such as a collection of generalized forces, or a row or column matrix considered from the viewpoint of linear algebra to belong to an n-dimensional tangent space, orthogonal space, etc.

In Ref. [26], a comprehensive, consistent, and concise method is established for identifying a set of forces needed to constrain the behavior of a mechanical system modeled as a set of particles and rigid bodies. The purpose of this paper is to apply the method to constraints described by nonholonomic equations that are inherently nonlinear in velocity. (It is to be understood that the term "velocity," used in the general case of a system of particles, subsumes "angular velocity" in the special case in which a subset of particles makes up a rigid body. The term "acceleration" likewise encompasses an angular counterpart.) An essential feature of the method consists of expressing constraint equations in vector form rather than entirely in terms of scalars and matrices as is customary. A constraint equation that has been differentiated once or twice with respect to time, so that it contains the acceleration of a point or the angular acceleration of a rigid body, is said to be written at the acceleration level. Likewise, a constraint equation at the velocity level is one that has been differentiated at most once, so that it contains the velocity of a point or the angular velocity of a rigid body. The method developed in Ref. [26] is applied in that work to configuration constraints, and to motion constraints expressed with equations that are linear in velocity when written at the velocity level. It so happens that the method can be applied whenever constraints can be described at the acceleration level by a set of independent equations that are linear in acceleration; therefore, it is applicable to constraint equations that are nonlinear in velocity when written at the velocity level.

The method in question yields expressions in vector form for constraint forces, and for torques of couples formed by constraint forces (hereafter referred to as constraint forces and torques). Thus, the directions of these vectors are identified, together with the specific point at which a constraint force must be applied, and the particular body upon which a constraint torque must be exerted. Such information about the vector quantities is of interest in its own right, and is to be preferred over the information contained in a matrix whose elements are scalar generalized constraint forces. In the process of constructing generalized constraint forces, information about the direction, magnitude, and point or body of application of constraint forces and torques becomes lost; in principle, each generalized constraint force is a sum of contributions from every constraint force and torque acting on a mechanical system. Although generalized constraint forces can be computed in a straightforward manner from knowledge of constraint forces and torques, usually it is impractical to invert the process and recover the original information about constraint forces and torques from generalized constraint forces.

Anderson is concerned in Ref. [27] with configuration constraints and motion constraints described by nonholonomic equations that are linear in the motion variables. Although such constraints are not the direct subject of the present investigation, Anderson makes an observation that is nevertheless relevant to our discussion. Often, a Lagrange multiplier or undetermined multiplier used to treat a constrained system is not related in a clear way to any particular constraint force or torque. In the method introduced here, each multiplier has a straightforward relationship to a constraint force or torque.

The emphasis in this paper is on analytic derivation of equations of motion that do or do not contain evidence of forces and torques needed to impose motion constraints described with inherently nonlinear nonholonomic equations. This stands in contrast to methods of computational dynamics, where the object is numerical formulation and solution of equations of motion. With knowledge of constraint forces and torques obtained by inspection of constraint equations written in vector form, and the two new approaches developed here, the analyst can form explicit equations of motion by hand or with the aid of symbolic algebra software. Equations that do not contain evidence of constraint forces can be formed directly; they need not be obtained from numerical manipulations of equations in which evidence of constraint forces is present.

The remainder of the paper is organized as follows. First, a treatment of nonlinear nonholonomic constraint equations is undertaken in Sec. 2 for a generic system of particles; the results are applicable whether or not a subset of particles makes up a rigid body. The method of Ref. [26] is used to identify directions of constraint forces and the particles to which they must be applied. The constraint forces are used together with extensions to Kane's method (Ref. [28]) to obtain two new ways of deriving dynamical equations of motion. The first of these is useful when one is interested in the time histories of the constraint forces; it produces dynamical equations that contain evidence of the constraint forces needed to satisfy the nonlinear nonholonomic constraint equations. On the other hand, the second approach can be used when one is not interested in the constraint forces but requires explicit dynamical equations governing the motion of the constrained system; constraint forces are not in evidence in the minimal equations of motion obtained with this approach. The novelty in the second case rests in the use of nonholonomic partial accelerations rather than the nonholonomic partial velocities employed in Kane's method. Both formulations are applied in Sec. 3 to an example in which the velocities of two particles must remain perpendicular. The resulting equations of motion are solved numerically. Constraint forces are identified in Sec. 4 for two other examples in which the velocities of two particles must either remain parallel, or equal in magnitude. In connection with Appell's particle, a constraint force is identified in Sec. 5; a second demonstration of the two approaches for obtaining equations of motion is performed, and the equations are compared to existing results. Finally, Sec. 6 contains the essential steps that must be taken to extend the ideas presented in Sec. 2 from a discussion in terms of a system of particles to the practical case in which a subset of the particles makes up a rigid body. Concluding remarks are supplied in Sec. 7.

2 Equations of Motion for Complex Nonholonomic Systems

It is instructive to recall that configuration constraints are, in general, expressed at the position level with nonlinear holonomic constraint equations. However, when these relationships are expressed at the velocity level they are linear in the velocity vectors or, what is the same, linear in the motion variables as shown in Ref. [26]. Similarly, motion constraints in general are described at the velocity level by nonlinear nonholonomic constraint equations but, when expressed at the acceleration level, they are linear in the acceleration vectors. In other words, when written in scalar form the latter relationships are linear in the time derivatives of motion variables. It is also important to remember that forces associated with configuration constraints are in evidence in equations of motion formed with partial velocities obtained from velocity expressions in which points are permitted to have certain velocities that they cannot in fact possess (Ref. [28]). These same forces are not in evidence when partial velocities are obtained from velocity expressions that account for the configuration constraints. Two important conclusions follow from these observations. First, forces needed to satisfy nonlinear nonholonomic constraint equations can be formed with the approach described in Ref. [26]. Second, forces needed to ensure satisfaction of inherently nonlinear nonholonomic constraint equations can be brought into, or left out of, evidence in equations of motion by making use of partial accelerations obtained from acceleration expressions that respectively do not, or do, account for the associated motion constraints.

Suppose that a simple nonholonomic system S (Ref. [28]) is made up of particles $P_1, \ldots, P_\nu$. The configuration of S in a Newtonian reference frame N is described by generalized coordinates $q_1, \ldots, q_n$, and the motion of S is characterized by independent motion variables $u_1, \ldots, u_p$. Suppose further that S is subject to ℓ nonlinear nonholonomic constraint equations

$$h_s({}^N\mathbf{v}^{P_1}, \ldots, {}^N\mathbf{v}^{P_\nu}, t) = 0 \qquad (s = 1, \ldots, \ell) \tag{1}$$

where ${}^N\mathbf{v}^{P_i}$ is the velocity of particle P_i $(i = 1, \ldots, \nu)$ in N, and where t denotes time. In this case S is referred to as a *complex nonholonomic system.* Differentiation of these relationships with respect to t in N yields

$$\sum_{i=1}^{\nu} {}^N\mathbf{a}^{P_i} \cdot \mathbf{W}_{is} + Z_s = 0 \qquad (s = 1, \ldots, \ell) \tag{2}$$

where $\mathbf{W}_{is}$ are vector functions of $q_1, \ldots, q_n, u_1, \ldots, u_p$ and t in N, and Z_s are scalar functions of the same variables. The acceleration of P_i in N is represented by ${}^N\mathbf{a}^{P_i}$. When these relationships are satisfied the motion variable time derivatives $\dot{u}_1, \ldots, \dot{u}_p$ are no longer independent, as discussed presently. According to Ref. [26] one can inspect these relationships and conclude that constraint forces are given by

$$\mathbf{C}_{is} = \lambda_s \mathbf{W}_{is} \qquad (i = 1, \ldots, \nu;\ s = 1, \ldots, \ell) \tag{3}$$

where λ_s are scalar multipliers whose time histories may be of interest. As discussed in Ref. [26], $\mathbf{C}_{is}$ is parallel to $\mathbf{W}_{is}$ and in general it must be applied to P_i in order to satisfy the constraint equations (2). The fundamental definition of Kane's generalized active forces involves the dot product of two vectors; $\mathbf{C}_{is}$ is one such vector. Knowledge of the direction and point of application of $\mathbf{C}_{is}$, and its relationship to λ_s, is important for its own sake. It is at least as important as having a collection of generalized constraint forces in hand, if not more so. The technique of inspecting Eqs. (2) systematically establishes the direction and point of application of a constraint force very soon after a constraint equation is available at the acceleration level in vector form, generally much sooner and with less labor than when working with constraint equations written entirely in terms of scalars and matrices.

Dynamical equations of motion to which $\mathbf{C}_{is}$ do contribute are given by

$$\begin{aligned} \tilde{F}_r + \tilde{F}_r^\star &= \sum_{i=1}^{\nu} {}^N\tilde{\mathbf{v}}_r^{P_i} \cdot \left(\mathbf{R}_i - m_i\, {}^N\mathbf{a}^{P_i}\right) \\ &= \sum_{i=1}^{\nu} {}^N\tilde{\mathbf{v}}_r^{P_i} \cdot \left(\mathbf{f}_i + \sum_{s=1}^{\ell} \lambda_s \mathbf{W}_{is} - m_i\, {}^N\mathbf{a}^{P_i}\right) = 0 \qquad (r = 1, \ldots, p) \end{aligned} \tag{4}$$

where $\tilde{F}_r$, $\tilde{F}_r^\star$, and ${}^N\tilde{\mathbf{v}}_r^{P_i}$ respectively denote the rth nonholonomic generalized active force for S in N, nonholonomic generalized inertia force for S in N, and nonholonomic partial velocity of P_i in N (Ref. [28]). The mass of P_i is indicated by m_i. The resultant $\mathbf{R}_i$ of all contact forces and distance forces acting on P_i is regarded as the sum of the constraint forces $\sum_{s=1}^{\ell} \lambda_s \mathbf{W}_{is}$ that must be applied to ensure satisfaction of Eqs. (2), added to the resultant of all other forces, $\mathbf{f}_i$. Equations (4) together with Eqs. (2) furnish the number of relationships needed to solve for the unknown quantities $\dot{u}_1, \ldots, \dot{u}_p, \lambda_1, \ldots, \lambda_\ell$. One employs these relationships if the time histories of $\lambda_1, \ldots, \lambda_\ell$ are of interest.

A reduced or minimal set of dynamical equations to which $\mathbf{C}_{is}$ do not contribute is given by

$$\tilde{\tilde{F}}_r + \tilde{\tilde{F}}_r^\star = \sum_{i=1}^{\nu} {}^N\tilde{\mathbf{a}}_r^{P_i} \cdot \left(\mathbf{f}_i + \sum_{s=1}^{\ell} \lambda_s \mathbf{W}_{is} - m_i\, {}^N\mathbf{a}^{P_i}\right)$$

$$= \sum_{i=1}^{\nu} {}^N\tilde{\mathbf{a}}_r^{P_i} \cdot \left(\mathbf{f}_i - m_i \, {}^N\mathbf{a}^{P_i} \right) = 0 \qquad (r = 1, \ldots, c) \tag{5}$$

where

$$c \triangleq p - \ell \tag{6}$$

is the number of degrees of freedom of S in N. When speaking of $\tilde{\tilde{F}}_r$ and $\tilde{\tilde{F}}_r^\star$ it is convenient to refer to them, respectively, as the rth nonholonomic generalized active force for S in N and the rth nonholonomic generalized inertia force for S in N, but the double tilde notation should be used to indicate they have been formed with ${}^N\tilde{\mathbf{a}}_r^{P_i}$, the rth *nonholonomic partial acceleration* of P_i in N, rather than ${}^N\tilde{\mathbf{v}}_r^{P_i}$. When one is not interested in time histories of $\lambda_1, \ldots, \lambda_\ell$, one can form Eqs. (5) directly by forming the dot products indicated in the second line rather than the first line. Moreover, Eqs. (4) need not be constructed first. Directly forming Eqs. (5) thus eliminates the need for assembling a constraint Jacobian and an orthogonal complement, and subsequently using the latter matrix to annihilate the former.

It is important to realize that the nonholonomic partial accelerations ${}^N\tilde{\mathbf{a}}_r^{P_i}$ in Eqs. (5) are distinct from the nonholonomic partial velocities ${}^N\tilde{\mathbf{v}}_r^{P_i}$ in Eqs. (4). In addition, one must have practical instructions for obtaining the vectors ${}^N\tilde{\mathbf{a}}_r^{P_i}$. Moreover, it is useful to understand the role played by the vectors ${}^N\tilde{\mathbf{a}}_r^{P_i}$ in making it unnecessary, in general, to include the multipliers in Eqs. (5).

The acceleration of P_i in N can be written uniquely in terms of $\dot{u}_1, \ldots, \dot{u}_p$,

$${}^N\mathbf{a}^{P_i} = \sum_{r=1}^{p} {}^N\mathbf{a}_r^{P_i} \, \dot{u}_r + {}^N\mathbf{a}_t^{P_i} \qquad (i = 1, \ldots, \nu) \tag{7}$$

and, also uniquely, in terms of the *independent* motion variable time derivatives $\dot{u}_1, \ldots, \dot{u}_c$,

$${}^N\mathbf{a}^{P_i} = \sum_{r=1}^{c} {}^N\tilde{\mathbf{a}}_r^{P_i} \, \dot{u}_r + {}^N\tilde{\mathbf{a}}_t^{P_i} \qquad (i = 1, \ldots, \nu) \tag{8}$$

Equations (7) and (8) are analogous to Eqs. (2.14.2) and (2.14.4) in Ref. [28], where it is established that a holonomic partial velocity $\mathbf{v}_r$ is distinct from a nonholonomic partial velocity $\tilde{\mathbf{v}}_r$. Similarly, the partial acceleration ${}^N\mathbf{a}_r^{P_i}$ is decidedly different from the nonholonomic partial acceleration ${}^N\tilde{\mathbf{a}}_r^{P_i}$ because the right hand member of Eqs. (8) involves only the independent motion variable time derivatives.

Equations (7) can be obtained from Eq. (2.14.4) of Ref. [28] by differentiation with respect to t in N, in which case the partial acceleration ${}^N\mathbf{a}_r^{P_i}$ is seen to be identical to the nonholonomic partial velocity of P_i in N,

$$ {}^N\mathbf{a}_r^{P_i} \triangleq {}^N\tilde{\mathbf{v}}_r^{P_i} \qquad (i = 1, \ldots, \nu;\ r = 1, \ldots, p) \tag{9} $$

and the acceleration remainder ${}^N\mathbf{a}_t^{P_i}$ is defined to be

$$ {}^N\mathbf{a}_t^{P_i} \triangleq \sum_{r=1}^{p} \left(\frac{{}^N d}{dt} {}^N\tilde{\mathbf{v}}_r^{P_i} \right) u_r + \frac{{}^N d}{dt} {}^N\tilde{\mathbf{v}}_t^{P_i} \qquad (i = 1, \ldots, \nu) \tag{10} $$

Substitution from Eqs. (7) into (2) gives

$$ \sum_{r=1}^{p} \left(\sum_{i=1}^{\nu} {}^N\mathbf{a}_r^{P_i} \cdot \mathbf{W}_{is} \right) \dot{u}_r + \sum_{i=1}^{\nu} {}^N\mathbf{a}_t^{P_i} \cdot \mathbf{W}_{is} + Z_s = 0 \qquad (s = 1, \ldots, \ell) \tag{11} $$

The coefficients of $\dot{u}_r$ and the remaining terms can be abbreviated respectively by means of two definitions,

$$ \alpha_{sr} \triangleq \sum_{i=1}^{\nu} {}^N\mathbf{a}_r^{P_i} \cdot \mathbf{W}_{is} \qquad (s = 1, \ldots, \ell;\ r = 1, \ldots, p) \tag{12} $$

and

$$ \gamma_s \triangleq Z_s + \sum_{i=1}^{\nu} {}^N\mathbf{a}_t^{P_i} \cdot \mathbf{W}_{is} \qquad (s = 1, \ldots, \ell) \tag{13} $$

where α_{sr} and γ_s are functions of $q_1, \ldots, q_n, u_1, \ldots, u_p$, and the time t. These definitions allow Eqs. (11) to be rewritten in a form that is linear in the time derivatives of the motion variables

$$ \sum_{r=1}^{p} \alpha_{sr} \dot{u}_r + \gamma_s = 0 \qquad (s = 1, \ldots, \ell) \tag{14} $$

These relationships express the dependence of ℓ time derivatives of the motion variables, say $\dot{u}_{c+1}, \ldots, \dot{u}_p$, on the remaining ones $\dot{u}_1, \ldots, \dot{u}_c$. It is assumed that these equations can in fact be solved for $\dot{u}_{c+1}, \ldots, \dot{u}_p$ in terms of $\dot{u}_1, \ldots, \dot{u}_c$. The dependent motion variable time derivatives are written in terms of the independent ones in a manner analogous to Eqs. (2.13.1) of Ref. [28],

$$ \dot{u}_{c+r} = \sum_{s=1}^{c} A_{rs} \dot{u}_s + B_r \qquad (r = 1, \ldots, \ell) \tag{15} $$

With a relationship for ${}^N\mathbf{a}^{P_i}$ in hand having the form of Eqs. (7), one simply embeds the acceleration level constraint equations by rewriting $\dot{u}_{c+1}, \ldots, \dot{u}_p$ in terms of $\dot{u}_1, \ldots, \dot{u}_c$

to obtain an expression in the form of Eqs. (8). Nonholonomic partial accelerations ${}^N\tilde{\mathbf{a}}_r^{P_i}$ are subsequently obtained in the same way as partial velocities, namely by inspecting the resulting relationship for acceleration to determine the vector coefficients of $\dot{u}_r$ for $r = 1, \ldots, c$.

When dealing with simple nonholonomic systems and the associated constraint equations (2.13.1) of Ref. [28], the analyst chooses which p of $u_1, \ldots, u_n$ to regard as independent; of course, the remaining motion variables are then regarded as dependent. The choice is made during the process of deriving explicit equations of motion. The same is true in the case of Eqs. (15) here; the analyst chooses which c of $\dot{u}_1, \ldots, \dot{u}_p$ are considered independent. In neither case is the decision based on the result of numerical procedures used in connection with the computational method of coordinate partitioning discussed in Refs. [29] and [30]. One does not, for instance, "take advantage of the numerical structure of the Jacobian matrix" (Ref. [30]). As Anderson notes in Ref. [27], coordinate partitioning is an iterative, computationally expensive procedure that cannot be used in explicit symbolic formulation of equations of motion.

The remainder of this section is devoted to accomplishing two goals. The first is to determine the contribution of the constraint forces $\mathbf{C}_{is}$ $(i = 1, \ldots, \nu; s = 1, \ldots, \ell)$ to the nonholonomic generalized active forces $\tilde{F}_r$ $(r = 1, \ldots, p)$. The second is to show, in general, that $\mathbf{C}_{is}$ contribute nothing to $\tilde{\tilde{F}}_r$ $(r = 1, \ldots, c)$.

Achieving the first objective is straightforward. Nonholonomic generalized active forces for S in N are defined by Eqs. (4.4.1) in Ref. [28] as the sum of dot products of pairs of vectors:

$$\tilde{F}_r \triangleq \sum_{i=1}^{\nu} {}^N\tilde{\mathbf{v}}_r^{P_i} \cdot \mathbf{R}_i \qquad (r = 1, \ldots, p) \tag{16}$$

Let $\mathbf{C}_i$ represent the resultant of the constraint forces $\mathbf{C}_{is}$ applied to P_i in order to ensure satisfaction of Eqs. (2), so that

$$\mathbf{C}_i \triangleq \sum_{s=1}^{\ell} \mathbf{C}_{is} = \sum_{s=1}^{\ell} \lambda_s \mathbf{W}_{is} \qquad (i = 1, \ldots, \nu) \tag{17}$$

The resultant $\mathbf{R}_i$ of all contact forces and distance forces acting on P_i can then be regarded as the sum of the constraint force, $\mathbf{C}_i$, and the resultant of all other forces, $\mathbf{f}_i$. Hence, $\tilde{F}_r$

is made up of contributions $(\tilde{F}_r)_\mathcal{C}$ from the constraint forces acting on S and $(\tilde{F}_r)_\mathcal{F}$ from all other forces acting on S,

$$\tilde{F}_r = (\tilde{F}_r)_\mathcal{C} + (\tilde{F}_r)_\mathcal{F} \triangleq \sum_{i=1}^{\nu} {}^N\tilde{\mathbf{v}}_r^{P_i} \cdot \mathbf{C}_i + \sum_{i=1}^{\nu} {}^N\tilde{\mathbf{v}}_r^{P_i} \cdot \mathbf{f}_i \qquad (r = 1, \ldots, p) \tag{18}$$

The contribution from the constraint forces can be singled out, and it is given by

$$(\tilde{F}_r)_\mathcal{C} = \sum_{i=1}^{\nu} {}^N\tilde{\mathbf{v}}_r^{P_i} \cdot \sum_{s=1}^{\ell} \lambda_s \mathbf{W}_{is} = \sum_{s=1}^{\ell} \lambda_s \alpha_{sr} \qquad (r = 1, \ldots, p) \tag{19}$$

where α_{sr} has the same meaning as in Eqs. (12). As is true when obtaining any generalized active force by using the techniques of Ref. [28], the recommended approach is to form the dot products indicated in Eqs. (16), (18), and (19).

It can now be shown that the constraint forces $\mathbf{C}_{is}$ make no contribution to any of $\tilde{\tilde{F}}_r$. What follows is a general proof of this result.

Equations (15) can be written in matrix form,

$$\dot{u}_D = A\dot{u}_I + B \tag{20}$$

where $\dot{u}_I$ is a $c \times 1$ column array containing the independent quantities $\dot{u}_1, \ldots, \dot{u}_c$, $\dot{u}_D$ is an $\ell \times 1$ column array containing the dependent quantities $\dot{u}_{c+1}, \ldots, \dot{u}_p$, A is an $\ell \times c$ matrix whose elements are A_{rs}, and B is an $\ell \times 1$ column array with elements B_r. Equations (14) can also be recast in matrix form as

$$\alpha_I \dot{u}_I + \alpha_D \dot{u}_D + \gamma = 0 \tag{21}$$

where α_I is an $\ell \times c$ matrix, α_D is an $\ell \times \ell$ matrix, and γ is an $\ell \times 1$ column array whose elements are $\gamma_1, \ldots, \gamma_\ell$. The motion variable time derivatives can always be ordered such that α_D has an inverse as long as the constraint equations are independent, thus

$$\dot{u}_D = -{\alpha_D}^{-1} \alpha_I \dot{u}_I - {\alpha_D}^{-1} \gamma \tag{22}$$

and comparison of this relationship with Eq. (20) produces the definitions

$$A \triangleq -{\alpha_D}^{-1} \alpha_I, \qquad B \triangleq -{\alpha_D}^{-1} \gamma \tag{23}$$

The remaining steps taken in the proof are not intended for use in constructing $\tilde{\tilde{F}}_r$; for this, one simply forms the dot products indicated in Eqs. (5) and the proven result is a natural consequence of carrying out the required operations. As has been pointed out, Eqs. (5) can be formed directly without first obtaining Eqs. (4).

Equations (19) can be expressed in matrix form as

$$(\tilde{F})_{\mathcal{C}} = \alpha^{\mathrm{T}}\lambda = \left\{ \begin{array}{c} {\alpha_I}^{\mathrm{T}}\lambda \\ {\alpha_D}^{\mathrm{T}}\lambda \end{array} \right\} \triangleq \left\{ \begin{array}{c} (\tilde{F}_I)_{\mathcal{C}} \\ (\tilde{F}_D)_{\mathcal{C}} \end{array} \right\} \tag{24}$$

where λ is an $\ell \times 1$ column array whose elements are $\lambda_1, \dots, \lambda_\ell$, $(\tilde{F}_I)_{\mathcal{C}}$ is a $c \times 1$ column array with elements $(\tilde{F}_1)_{\mathcal{C}}, \dots, (\tilde{F}_c)_{\mathcal{C}}$, and $(\tilde{F}_D)_{\mathcal{C}}$ is an $\ell \times 1$ column array with elements $(\tilde{F}_{c+1})_{\mathcal{C}}, \dots, (\tilde{F}_p)_{\mathcal{C}}$. In view of the analogous relationship between Eqs. (2.13.1) of Ref. [28] and Eqs. (15), one can write a relationship analogous to Eqs. (4.4.3) in Ref. [28],

$$(\tilde{\tilde{F}}_r)_{\mathcal{C}} = (\tilde{F}_r)_{\mathcal{C}} + \sum_{s=1}^{\ell} (\tilde{F}_{c+s})_{\mathcal{C}}\ A_{sr} \qquad (r = 1, \dots, c) \tag{25}$$

These relationships can be expressed in matrix form as

$$\begin{aligned} (\tilde{\tilde{F}})_{\mathcal{C}} &= (\tilde{F}_I)_{\mathcal{C}} + A^{\mathrm{T}}(\tilde{F}_D)_{\mathcal{C}} \\ &= {\alpha_I}^{\mathrm{T}}\lambda + A^{\mathrm{T}}{\alpha_D}^{\mathrm{T}}\lambda = \left({\alpha_I}^{\mathrm{T}} + A^{\mathrm{T}}{\alpha_D}^{\mathrm{T}}\right)\lambda \end{aligned} \tag{26}$$

The term in the parenthesis is observed to vanish by noting

$$0 = \alpha_I - \alpha_I = \alpha_I - \alpha_D {\alpha_D}^{-1}\alpha_I = \alpha_I + \alpha_D A \tag{27}$$

Hence, the transpose of this relationship is ${\alpha_I}^{\mathrm{T}} + A^{\mathrm{T}}{\alpha_D}^{\mathrm{T}} = 0$. This step may be viewed as premultiplication of α^{T} by an orthogonal complement matrix $[I_c\ A^{\mathrm{T}}]$, where I_c is the $c \times c$ identity matrix. In practice, direct construction of Eqs. (5) does not require creation or use of an orthogonal complement. In any event, it is shown that

$$(\tilde{\tilde{F}}_r)_{\mathcal{C}} = 0 \qquad (r = 1, \dots, c) \tag{28}$$

or, in words, motion constraints described by inherently nonlinear nonholonomic constraint equations require the application of forces that make no contributions to any of the nonholonomic generalized active forces $\tilde{\tilde{F}}_r$. Because these contributions are defined in terms of nonholonomic partial accelerations ${}^N\tilde{\mathbf{a}}_r^{P_i}$ as

$$(\widetilde{\widetilde{F}}_r)_{\mathcal{C}} = \sum_{i=1}^{\nu} {}^N\tilde{\mathbf{a}}_r^{P_i} \cdot \mathbf{C}_i = \sum_{i=1}^{\nu} {}^N\tilde{\mathbf{a}}_r^{P_i} \cdot \sum_{s=1}^{\ell} \lambda_s \mathbf{W}_{is} = \sum_{s=1}^{\ell} \lambda_s \sum_{i=1}^{\nu} {}^N\tilde{\mathbf{a}}_r^{P_i} \cdot \mathbf{W}_{is} \qquad (r = 1, \ldots, c) \tag{29}$$

it can be concluded that

$$\sum_{i=1}^{\nu} {}^N\tilde{\mathbf{a}}_r^{P_i} \cdot \mathbf{W}_{is} = 0 \qquad (r = 1, \ldots, c;\ s = 1, \ldots, \ell) \tag{30}$$

The utility of this proof lies in the general demonstration that, when directly forming Eqs. (5) for a particular system, the constraint forces $\mathbf{C}_i$ may be included in $\mathbf{R}_i$ or they may be omitted; in either case they will not contribute to $\widetilde{\widetilde{F}}_r$.

3 Two Particles with Perpendicular Velocities

An example is provided to illustrate application of Eqs. (4) and (5) to form equations of motion in which constraint forces respectively are and are not in evidence. A system of two individual particles is subject to a requirement that the velocity in a Newtonian reference frame N of one particle must remain perpendicular to the velocity in N of the other particle. The associated nonholonomic constraint equation is inherently nonlinear. Implementation of the constraint would require the sort of computations that are associated with a control system, as well as ideal actuators and sensors; thus, the example features a servo-constraint. The demonstration is followed by discussion of a similar example from the literature in which the constraint is imposed by purely mechanical means, and it is shown that the nonholonomic constraint equation can in that case be expressed as a linear relationship.

Two pucks moving on an air-bearing table fixed in a Newtonian reference frame N are modeled as particles P_1 with a mass of m_1, and P_2 with a mass of m_2. Let two orthogonal unit vectors $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_2$ be fixed in N and define the plane of the table, and let unit vector $\hat{\mathbf{n}}_3 \triangleq \hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2$ be normal to the plane. An external force $\mathbf{f}_1 = \sigma_1 \hat{\mathbf{n}}_1 + \sigma_2 \hat{\mathbf{n}}_2$ is applied to P_1 whereas a force $\mathbf{f}_2 = \sigma_3 \hat{\mathbf{n}}_1 + \sigma_4 \hat{\mathbf{n}}_2$ is applied to P_2. The motion of this system is regarded as unconstrained. Suppose that the velocities ${}^N\mathbf{v}^{P_1}$ and ${}^N\mathbf{v}^{P_2}$ of P_1 and P_2 in N are to be constrained such that they must remain perpendicular at all times. Let $m_1 = 1$ kg, $m_2 = 2$

kg, and let $\mathbf{f}_1$ and $\mathbf{f}_2$ be characterized by the constants $\sigma_1 = 1.0$ N, $\sigma_2 = 0$ N, $\sigma_3 = 1.0$ N, and $\sigma_4 = 0$ N. At $t = 0$ the velocities of P_1 and P_2 in N are given by ${}^N\mathbf{v}^{P_1} = 0.3\hat{\mathbf{n}}_1 + 0.4\hat{\mathbf{n}}_2$ m/s, and ${}^N\mathbf{v}^{P_2} = 0.4\hat{\mathbf{n}}_1 - 0.3\hat{\mathbf{n}}_2$ m/s. The initial position vectors $\mathbf{p}_i$ from a point O fixed in N to P_i are given by $\mathbf{p}_1 = 1\hat{\mathbf{n}}_1 - 2\hat{\mathbf{n}}_2$ m, and $\mathbf{p}_2 = 1\hat{\mathbf{n}}_1 + 2\hat{\mathbf{n}}_2$ m.

First, a constraint equation is written at the acceleration level in vector form. It is inspected to construct expressions for the constraint forces that must be applied to P_1 and P_2 in order for the constraint to be obeyed. A constraint force can be applied to a puck, for example, by four orthogonally mounted thrusters. Equations (4) are then employed to produce dynamical equations of motion in which the constraint forces play a part, and these equations are solved numerically together with kinematical differential equations.

The constraint can be expressed by the relationship

$$ {}^N\mathbf{v}^{P_2} \cdot {}^N\mathbf{v}^{P_1} = 0 \tag{31} $$

This constraint equation is nonlinear in the velocity vectors because more than one velocity appears in a dot product; it is also nonlinear in motion variables, as will become apparent. Differentiation with respect to t in N brings the constraint equation to the acceleration level, where it is seen to be linear in the acceleration vectors because only one such vector appears in each dot product.

$$ {}^N\mathbf{a}^{P_2} \cdot {}^N\mathbf{v}^{P_1} + {}^N\mathbf{a}^{P_1} \cdot {}^N\mathbf{v}^{P_2} = 0 \tag{32} $$

With Eqs. (2) and (3) in mind, it can be concluded that the constraint requires application of the forces

$$ \mathbf{C}_2 = \lambda\, {}^N\mathbf{v}^{P_1}, \qquad \mathbf{C}_1 = \lambda\, {}^N\mathbf{v}^{P_2} \tag{33} $$

to P_2 and P_1 respectively. The constraint forces $\mathbf{C}_1$ and $\mathbf{C}_2$ need not be of equal magnitudes because the constraint does not require ${}^N\mathbf{v}^{P_2}$ and ${}^N\mathbf{v}^{P_1}$ to be equal in magnitude. The constraint force $\mathbf{C}_1$ is perpendicular to $\mathbf{C}_2$ when the constraint is satisfied. All of this valuable information concerning the vectors $\mathbf{C}_1$ and $\mathbf{C}_2$, including their relationship to λ, is obtained by inspecting Eqs. (32) rather than by attempting to infer it from examination of generalized constraint forces. The vector forms in Eqs. (33) are required for forming the dot products indicated in Eqs. (4).

The unconstrained system possesses four degrees of freedom in N, thus the motion can be characterized by four motion variables defined operationally as

$$ {}^{N}\mathbf{v}^{P_1} = u_1\hat{\mathbf{n}}_1 + u_2\hat{\mathbf{n}}_2, \qquad {}^{N}\mathbf{v}^{P_2} = u_3\hat{\mathbf{n}}_1 + u_4\hat{\mathbf{n}}_2 \tag{34} $$

These relationships are inspected to identify the vector coefficients of u_1, u_2, u_3, and u_4; that is, the nonholonomic partial velocities

$$ {}^{N}\tilde{\mathbf{v}}_1^{P_1} = \hat{\mathbf{n}}_1, \qquad {}^{N}\tilde{\mathbf{v}}_2^{P_1} = \hat{\mathbf{n}}_2, \qquad {}^{N}\tilde{\mathbf{v}}_3^{P_1} = \mathbf{0}, \qquad {}^{N}\tilde{\mathbf{v}}_4^{P_1} = \mathbf{0} \tag{35} $$

$$ {}^{N}\tilde{\mathbf{v}}_1^{P_2} = \mathbf{0}, \qquad {}^{N}\tilde{\mathbf{v}}_2^{P_2} = \mathbf{0}, \qquad {}^{N}\tilde{\mathbf{v}}_3^{P_2} = \hat{\mathbf{n}}_1, \qquad {}^{N}\tilde{\mathbf{v}}_4^{P_2} = \hat{\mathbf{n}}_2 \tag{36} $$

The partial velocities are referred to as nonholonomic, and the notation ${}^{N}\tilde{\mathbf{v}}_r^{P_i}$ is used to indicate that the expressions in Eqs. (34) would have accounted for any nonholonomic constraint equations *linear* in the motion variables, had they been called for. Dynamical equations of motion formed according to Eqs. (4) are readily written as

$$ m_1\dot{u}_1 = \sigma_1 + \lambda u_3, \quad m_1\dot{u}_2 = \sigma_2 + \lambda u_4, \quad m_2\dot{u}_3 = \sigma_3 + \lambda u_1, \quad m_2\dot{u}_4 = \sigma_4 + \lambda u_2 \tag{37} $$

The constraint equation expressed at the velocity level in vector form by Eq. (31) becomes, in scalar form,

$$ u_1 u_3 + u_2 u_4 = 0 \tag{38} $$

This relationship is nonlinear in the motion variables; hence, the constrained system cannot be treated as a holonomic system or even a simple nonholonomic system. The apparatus of Ref. [28] contains no provisions whatsoever for dealing with such a constraint equation, therefore it cannot be used to form the familiar holonomic or nonholonomic partial velocities. It is for this reason that nonholonomic partial accelerations are introduced in this paper; these vectors can be used to construct equations of motion devoid of λ, as is demonstrated shortly. Now, the constraint equation at the acceleration level is linear in the time derivatives of the motion variables,

$$ u_3\dot{u}_1 + u_4\dot{u}_2 + u_1\dot{u}_3 + u_2\dot{u}_4 = 0 \tag{39} $$

An analytical solution of the linear system of equations (37) and (39) for the five unknowns is manageable, and is given by

$$ \lambda = -\frac{m_1(\sigma_3 u_1 + \sigma_4 u_2) + m_2(\sigma_1 u_3 + \sigma_2 u_4)}{m_1({u_1}^2 + {u_2}^2) + m_2({u_3}^2 + {u_4}^2)} \tag{40} $$

$$\dot{u}_1 = \frac{\sigma_1 + \lambda u_3}{m_1}, \quad \dot{u}_2 = \frac{\sigma_2 + \lambda u_4}{m_1}, \quad \dot{u}_3 = \frac{\sigma_3 + \lambda u_1}{m_2}, \quad \dot{u}_4 = \frac{\sigma_4 + \lambda u_2}{m_2} \tag{41}$$

The configuration of P_1 and P_2 in N is described by four generalized coordinates introduced operationally as

$$\mathbf{p}_1 = q_1\hat{\mathbf{n}}_1 + q_2\hat{\mathbf{n}}_2, \qquad \mathbf{p}_2 = q_3\hat{\mathbf{n}}_1 + q_4\hat{\mathbf{n}}_2 \tag{42}$$

Four kinematical differential equations are given simply by

$$\dot{q}_r = u_r \qquad (r = 1, 2, 3, 4) \tag{43}$$

The dynamical and kinematical differential equations are integrated numerically with a variable step-size algorithm, using an absolute error of 1×10^{-8} and a relative error of 1×10^{-7}. The unconstrained trajectories ($\lambda = 0$) of P_1 and P_2 are displayed in the upper left of Fig. 1, to be compared to the constrained trajectories shown in the upper right. It is clear that ${}^N\mathbf{v}^{P_1}$ and ${}^N\mathbf{v}^{P_2}$ are becoming parallel in the absence of constraint forces, whereas they remain perpendicular when $\mathbf{C}_1$ and $\mathbf{C}_2$ are applied. A time history of λ is shown in the lower left of Fig. 1. The constraint requires ${}^N\mathbf{v}^{P_2}$ to remain perpendicular to ${}^N\mathbf{v}^{P_1}$; hence, the cosine of the angle between the two vectors calculated as $\cos\theta = {}^N\mathbf{v}^{P_2} \cdot {}^N\mathbf{v}^{P_1}/(|{}^N\mathbf{v}^{P_2}||{}^N\mathbf{v}^{P_1}|)$, which should be 0, can be used as a measure of the failure of the numerical solution to satisfy the constraint. As seen in the lower right of Fig. 1, the solution meets the constraint very well.

One can virtually eliminate the small error evident in the time history of $\cos\theta$, and obtain dynamical equations of motion in which λ does not appear, by appealing directly to Eqs. (5). First, the accelerations in N of P_1 and P_2 are expressed as

$${}^N\mathbf{a}^{P_1} = \dot{u}_1\hat{\mathbf{n}}_1 + \dot{u}_2\hat{\mathbf{n}}_2, \qquad {}^N\mathbf{a}^{P_2} = \dot{u}_3\hat{\mathbf{n}}_1 + \dot{u}_4\hat{\mathbf{n}}_2 \tag{44}$$

The motion variable time derivatives $\dot{u}_1$, $\dot{u}_2$, and $\dot{u}_3$ can be chosen as independent. This leaves $\dot{u}_4$ as dependent, and one then substitutes from Eq. (39) to arrive at

$${}^N\mathbf{a}^{P_1} = \dot{u}_1\hat{\mathbf{n}}_1 + \dot{u}_2\hat{\mathbf{n}}_2, \qquad {}^N\mathbf{a}^{P_2} = \dot{u}_3\hat{\mathbf{n}}_1 - \frac{1}{u_2}(u_3\dot{u}_1 + u_4\dot{u}_2 + u_1\dot{u}_3)\hat{\mathbf{n}}_2 \tag{45}$$

The nonholonomic partial accelerations of P_1 and P_2 in N are identified as the vector coefficients of $\dot{u}_1$, $\dot{u}_2$, and $\dot{u}_3$,

$${}^N\tilde{\mathbf{a}}_1^{P_1} = \hat{\mathbf{n}}_1, \qquad {}^N\tilde{\mathbf{a}}_2^{P_1} = \hat{\mathbf{n}}_2, \qquad {}^N\tilde{\mathbf{a}}_3^{P_1} = \mathbf{0} \tag{46}$$

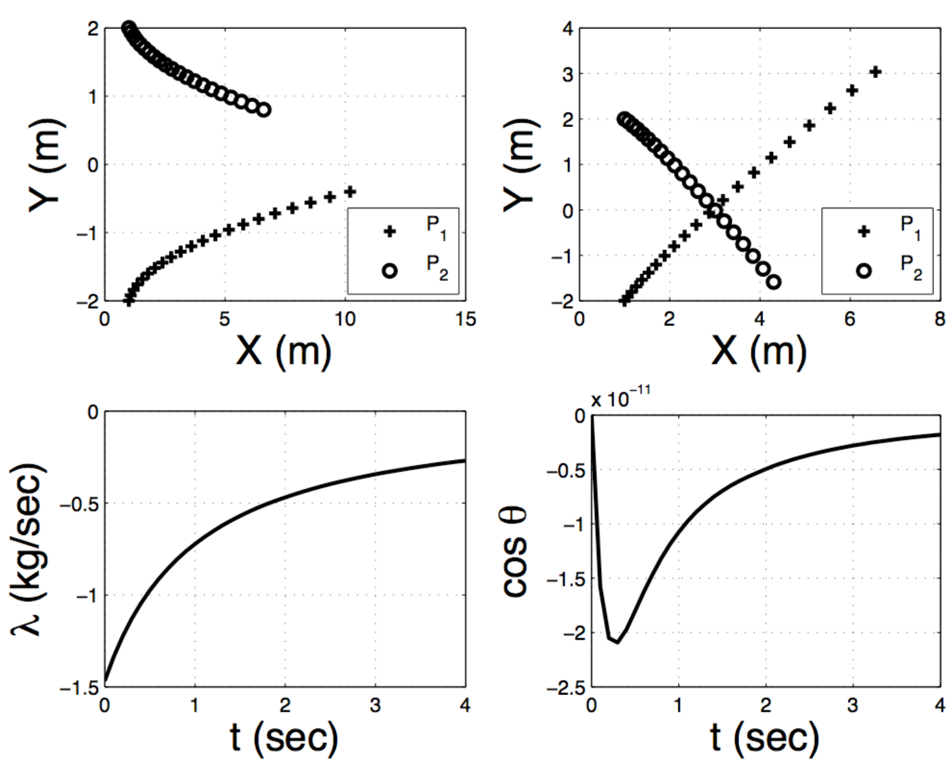

Figure 1: Two Particles with Perpendicular Velocities

$$ {}^{N}\tilde{\mathbf{a}}_1^{P_2} = -\frac{u_3}{u_2}\hat{\mathbf{n}}_2, \qquad {}^{N}\tilde{\mathbf{a}}_2^{P_2} = -\frac{u_4}{u_2}\hat{\mathbf{n}}_2, \qquad {}^{N}\tilde{\mathbf{a}}_3^{P_2} = \hat{\mathbf{n}}_1 - \frac{u_1}{u_2}\hat{\mathbf{n}}_2 \tag{47} $$

These vectors are evidently fewer in number than, and distinct from the nonholonomic partial velocities in Eqs. (35) and (36). Once they are in hand, nonholonomic generalized active forces for S in N can be formed according to the expressions

$$ \tilde{\tilde{F}}_r = {}^{N}\tilde{\mathbf{a}}_r^{P_1} \cdot (\mathbf{f}_1 + \lambda\, {}^{N}\mathbf{v}^{P_2}) + {}^{N}\tilde{\mathbf{a}}_r^{P_2} \cdot (\mathbf{f}_2 + \lambda\, {}^{N}\mathbf{v}^{P_1}) \qquad (r = 1, 2, 3) \tag{48} $$

The first of these is given by

$$ \begin{aligned} \tilde{\tilde{F}}_1 &= \hat{\mathbf{n}}_1 \cdot (\mathbf{f}_1 + \lambda\, {}^{N}\mathbf{v}^{P_2}) - \frac{u_3}{u_2}\hat{\mathbf{n}}_2 \cdot (\mathbf{f}_2 + \lambda\, {}^{N}\mathbf{v}^{P_1}) \\ &= \sigma_1 + \lambda u_3 - \frac{u_3}{u_2}(\sigma_4 + \lambda u_2) \\ &= \sigma_1 - \frac{u_3}{u_2}\sigma_4 \end{aligned} \tag{49} $$

Similarly,

$$ \tilde{\tilde{F}}_2 = \sigma_2 - \frac{u_4}{u_2}\sigma_4 \tag{50} $$

$$\tilde{\tilde{F}}_3 = \sigma_3 - \frac{u_1}{u_2}\sigma_4 \tag{51}$$

The multiplier λ is clearly absent from $\tilde{\tilde{F}}_1$, $\tilde{\tilde{F}}_2$, and $\tilde{\tilde{F}}_3$, and thus the constraint forces $\mathbf{C}_1$ and $\mathbf{C}_2$ do not contribute to the equations of motion. Considering the result proved in Sec. 2, one would be justified in omitting $\mathbf{C}_1$ and $\mathbf{C}_2$ from Eqs. (48) and (49), and thereby reducing the labor involved in forming dot products. The nonholonomic generalized active forces are obtained without first constructing relationships according to Eqs. (4), and without forming a Jacobian matrix or its orthogonal complement, and multiplying them together.

Nonholonomic generalized inertia forces are given by

$$\tilde{\tilde{F}}_r^\star = {}^N\tilde{\mathbf{a}}_r^{P_1} \cdot (-m_1\, {}^N\mathbf{a}^{P_1}) + {}^N\tilde{\mathbf{a}}_r^{P_2} \cdot (-m_2\, {}^N\mathbf{a}^{P_2}) \qquad (r = 1, 2, 3) \tag{52}$$

or

$$\begin{aligned}\tilde{\tilde{F}}_1^\star &= -m_1\dot{u}_1 - m_2\frac{u_3}{{u_2}^2}(u_3\dot{u}_1 + u_4\dot{u}_2 + u_1\dot{u}_3) \\ &= -\left[m_1 + m_2\left(\frac{u_3}{u_2}\right)^2\right]\dot{u}_1 - m_2\frac{u_3u_4}{{u_2}^2}\dot{u}_2 - m_2\frac{u_1u_3}{{u_2}^2}\dot{u}_3\end{aligned} \tag{53}$$

$$\begin{aligned}\tilde{\tilde{F}}_2^\star &= -m_1\dot{u}_2 - m_2\frac{u_4}{{u_2}^2}(u_3\dot{u}_1 + u_4\dot{u}_2 + u_1\dot{u}_3) \\ &= -m_2\frac{u_3u_4}{{u_2}^2}\dot{u}_1 - \left[m_1 + m_2\left(\frac{u_4}{u_2}\right)^2\right]\dot{u}_2 - m_2\frac{u_1u_4}{{u_2}^2}\dot{u}_3\end{aligned} \tag{54}$$

$$\begin{aligned}\tilde{\tilde{F}}_3^\star &= -m_2\dot{u}_3 - m_2\frac{u_1}{{u_2}^2}(u_3\dot{u}_1 + u_4\dot{u}_2 + u_1\dot{u}_3) \\ &= -m_2\frac{u_1u_3}{{u_2}^2}\dot{u}_1 - m_2\frac{u_1u_4}{{u_2}^2}\dot{u}_2 - m_2\left[1 + \left(\frac{u_1}{u_2}\right)^2\right]\dot{u}_3\end{aligned} \tag{55}$$

The mass matrix associated with these equations of motion is symmetric. After expressing u_4 as $-u_1u_3/u_2$ as required by Eq. (38), the dynamical equations of motion $\tilde{\tilde{F}}_r + \tilde{\tilde{F}}_r^\star = 0$ $(r = 1, 2, 3)$ and the kinematical differential equations (43) are integrated numerically using the initial conditions given in the problem statement. The paths of P_1 and P_2 are identical to those shown in the upper right plot of Fig. 1, and the absolute value of $\cos\theta$ remains less than 7.64×10^{-17} throughout the simulation.

In Refs. [13] and [14] Zekovich provides examples in which velocities of two particles are to remain perpendicular to one another. However, an additional configuration constraint is imposed on P_1 and P_2; they are connected by a “fork” that allows relative translation along the line joining P_1 and P_2. In other words, P_1 is regarded as fixed in a rigid body B, and a prismatic joint makes it possible for P_2 to move on B. A relationship having the form of Eq. (38) is given, and put forth as an example of a nonlinear nonholonomic constraint equation. However, the nonlinearity is contrived. The development in Ref. [13] is greatly simplified by working with a set of motion variables to be defined presently; furthermore, they are used to show that the relevant nonholonomic constraint equations can be written as linear expressions.

Let perpendicular unit vectors $\hat{\mathbf{b}}_1$ and $\hat{\mathbf{b}}_2$ be fixed in B such that they lie in the plane of motion of P_1 and P_2, and $\hat{\mathbf{b}}_1$ is in the direction of the prismatic joint that permits P_2 to slide on B. Unit vector $\hat{\mathbf{b}}_3$ is perpendicular to $\hat{\mathbf{b}}_1$ and $\hat{\mathbf{b}}_2$, and to the plane of the motion. Four motion variables are introduced operationally by writing ${}^N\mathbf{v}^{P_1} = u_1\hat{\mathbf{b}}_1 + u_2\hat{\mathbf{b}}_2$, ${}^N\boldsymbol{\omega}^B = u_3\hat{\mathbf{b}}_3$, and ${}^B\mathbf{v}^{P_2} = u_4\hat{\mathbf{b}}_1$. The angular velocity of B in N is denoted by ${}^N\boldsymbol{\omega}^B$, and the velocity of P_2 in B is indicated by ${}^B\mathbf{v}^{P_2}$. Hence, ${}^N\mathbf{v}^{P_2} = (u_1 + u_4)\hat{\mathbf{b}}_1 + (u_2 + q_4u_3)\hat{\mathbf{b}}_2$, where q_4 is the distance between P_1 and P_2. The perpendicular velocity constraint is expressed as ${}^N\mathbf{v}^{P_2} \cdot {}^N\mathbf{v}^{P_1} = u_1(u_1 + u_4) + u_2(u_2 + q_4u_3) = 0$.

Zekovich begins the analysis by attaching a sharp-edged circular disk, or blade, at P_1 with the edge perpendicular to $\hat{\mathbf{b}}_1$; the resulting constraint is expressed linearly as ${}^N\mathbf{v}^{P_1} \cdot \hat{\mathbf{b}}_1 = u_1 = 0$, and the corresponding Eq. (8) in Ref. [13] is likewise linear. With $u_1 = 0$, the velocity constraint is rewritten as ${}^N\mathbf{v}^{P_2} \cdot {}^N\mathbf{v}^{P_1} = u_2(u_2 + q_4u_3) = 0$, which corresponds to Eq. (9) of Ref. [13]. Zekovich then notes the constraint can be satisfied in either of two ways. The first possibility is imposition of the constraint expressed by the linear equation ${}^N\mathbf{v}^{P_1} \cdot \hat{\mathbf{b}}_2 = u_2 = 0$, in which case P_1 is fixed in N and the blade at P_1 is no longer necessary. The second possibility also involves a constraint described by a linear relationship ${}^N\mathbf{v}^{P_2} \cdot \hat{\mathbf{b}}_2 = u_2 + q_4u_3 = 0$; such a restriction can be imposed by fixing a blade at P_2 with the edge orthogonal to $\hat{\mathbf{b}}_2$. The presence of perpendicular constraint forces exerted by perpendicular blades is in keeping with the result of Eqs. (33), although it contradicts the direction of $\mathbf{R}_2$ indicated in Fig. 3a of Ref. [13].

4 Other Examples

Other restrictions on the motion of two separate particles give rise to nonholonomic constraint equations that are inherently nonlinear. Constraint forces required to ensure that the velocities in N of the two particles remain parallel, or equal in magnitude, are discussed briefly. This is followed with a mention of two examples involving a single particle.

First consider the requirement that ${}^N\mathbf{v}^{P_1}$ and ${}^N\mathbf{v}^{P_2}$ be parallel to each other. The constraint can be expressed as follows. The vector $\hat{\mathbf{n}}_3 \times {}^N\mathbf{v}^{P_1}$ is perpendicular to $\hat{\mathbf{n}}_3$ and to ${}^N\mathbf{v}^{P_1}$ by construction; therefore, requiring ${}^N\mathbf{v}^{P_2}$ to be parallel to ${}^N\mathbf{v}^{P_1}$ is the same as requiring

$$ {}^N\mathbf{v}^{P_2} \cdot (\hat{\mathbf{n}}_3 \times {}^N\mathbf{v}^{P_1}) = 0 \tag{56} $$

This constraint equation is observed to be nonlinear in the velocity vectors because more than one velocity appears in a dot product. Differentiation with respect to t in N brings the constraint equation to the acceleration level, where it is seen to be linear in the acceleration vectors.

$$ {}^N\mathbf{a}^{P_2} \cdot (\hat{\mathbf{n}}_3 \times {}^N\mathbf{v}^{P_1}) - {}^N\mathbf{a}^{P_1} \cdot (\hat{\mathbf{n}}_3 \times {}^N\mathbf{v}^{P_2}) = 0 \tag{57} $$

In view of Eqs. (2) and (3), the constraint requires application of the forces

$$ \mathbf{C}_2 = \lambda(\hat{\mathbf{n}}_3 \times {}^N\mathbf{v}^{P_1}), \qquad \mathbf{C}_1 = -\lambda(\hat{\mathbf{n}}_3 \times {}^N\mathbf{v}^{P_2}) \tag{58} $$

to P_2 and P_1 respectively. The constraint forces $\mathbf{C}_1$ and $\mathbf{C}_2$ need not be of equal magnitudes because the constraint does not require ${}^N\mathbf{v}^{P_2}$ and ${}^N\mathbf{v}^{P_1}$ to be equal in magnitude. Moreover, $\mathbf{C}_1$ and $\mathbf{C}_2$ may have the same direction or opposite directions depending on whether the directions of ${}^N\mathbf{v}^{P_1}$ and ${}^N\mathbf{v}^{P_2}$ are opposite or the same. As is the case in the example in Sec. 3, important information about constraint forces is obtained by inspecting a constraint equation written at the acceleration level in vector form. Extracting the same information from generalized constraint forces would be significantly more arduous. The relationship between the multiplier and the two constraint forces is clear-cut.

The first example in Refs. [13] and [14] is similar to the preceding situation, but an additional configuration constraint is imposed on P_1 and P_2; they are connected by a rod of fixed length $2L$. It is said that the requirement of parallel velocities can be achieved in practice by attaching at the rod's midpoint a blade that is perpendicular to the rod. A relationship is given with the form of Eq. (56) written entirely in terms of scalars, and offered as an example of a nonlinear nonholonomic constraint equation. However, in this instance the nonlinearity is contrived because the constraint dictated by the blade can in fact be described by a linear nonholonomic constraint equation. There appears to be some recognition of this in Ref. [13]. The directions of the constraint forces obtained in Eqs. (58) are seen to be the same as those indicated in the diagram on the right side of Fig. 2a in Ref. [13].

Next, suppose that ${}^N\mathbf{v}^{P_1}$ and ${}^N\mathbf{v}^{P_2}$ are required to have equal magnitudes rather than parallel directions or perpendicular directions. The constraint can be expressed by the relationship

$$ {}^N\mathbf{v}^{P_2} \cdot {}^N\mathbf{v}^{P_2} - {}^N\mathbf{v}^{P_1} \cdot {}^N\mathbf{v}^{P_1} = 0 \tag{59} $$

which is nonlinear in the velocity vectors. The acceleration level of the constraint equation is linear in the acceleration vectors,

$$ {}^N\mathbf{a}^{P_2} \cdot {}^N\mathbf{v}^{P_2} - {}^N\mathbf{a}^{P_1} \cdot {}^N\mathbf{v}^{P_1} = 0 \tag{60} $$

According to Eqs. (2) and (3), the constraint requires application of the forces

$$ \mathbf{C}_2 = \lambda\, {}^N\mathbf{v}^{P_2}, \qquad \mathbf{C}_1 = -\lambda\, {}^N\mathbf{v}^{P_1} \tag{61} $$

to P_2 and P_1 respectively. It is seen that $\mathbf{C}_1$ and $\mathbf{C}_2$ have equal magnitudes when the constraint is obeyed. Again, constraint force information is obtained by inspecting a constraint equation in vector form rather than by examining a collection of scalar generalized constraint forces, and the relationship of the multiplier to the constraint forces is completely evident.

The second example in Ref. [13] involves two particles whose velocities are to remain equal in magnitude; however, an additional configuration constraint is imposed on P_1 and P_2 as they are connected by a rod of fixed length. Zekovich observes the velocities are made equal in magnitude by placing a blade at the rod's midpoint and making the edge parallel to the rod. An expression having the same form as Eq. (59), written entirely with scalars, is offered as a nonlinear nonholonomic constraint equation. As is the case with Zekovich's first example, the nonlinearity is contrived and it can easily be shown that a linear nonholonomic constraint equation describes the constraint dictated by the blade. The diagram on the right side of Fig. 2b in Ref. [13] shows a constraint force in the direction of ${}^N\mathbf{v}^{P_1}$ and the other constraint force in the direction opposite to ${}^N\mathbf{v}^{P_2}$; this result can be made to agree with Eqs. (61) by renaming the two particles.

Jankowski has developed an approach for dealing with constraint equations that are not necessarily linear in acceleration. A procedure is set forth in Ref. [21] for forming dynamical equations of motion in which Lagrange multipliers do appear, and then the multipliers are eliminated by employing an orthogonal complement matrix to obtain a reduced set of equations. As mentioned previously, the paper concludes with an example involving a single particle P. It is readily demonstrated that Eqs. (4) and (5) can be used to obtain the results reported in Ref. [21] when the magnitude of the velocity ${}^N\mathbf{v}^P$ of P in N must have a prescribed time history; that is, ${}^N\mathbf{v}^P \cdot {}^N\mathbf{v}^P - v(t)^2 = 0$. Moreover, inspection of this constraint equation at the acceleration level indicates the constraint force applied to P is in the direction of ${}^N\mathbf{v}^P$, and Jankowski reaches the same conclusion. However, Eqs. (4) and (5) are not applicable to the subsequent case in which the magnitude of the acceleration ${}^N\mathbf{a}^P$ of P in N is a prescribed function of the time t, ${}^N\mathbf{a}^P \cdot {}^N\mathbf{a}^P - a(t)^2 = 0$

5 Appell's Particle

As mentioned earlier, the literature contains ample discussion of an example proposed by Appell in which a single particle must move in a uniform gravitational field so as to satisfy an inherently nonlinear nonholonomic constraint equation. A constraint force is identified in connection with this example, and a final brief demonstration of the use of Eqs. (4) and (5) shows that they lead to results obtained by Smith[1] and Van Dooren (Ref. [23]).

Three motion variables u_1, u_2, and u_3 are introduced such that the velocity ${}^N\mathbf{v}^P$ in a

[1] C. V. Smith, Jr., "Comments on Geometric Constraints, Virtual Displacements, and Ideal Constraint Forces," private communication, Sept. 25, 2002.

Newtonian reference frame N of a particle P is written as

$$ {}^N\mathbf{v}^P = u_1\hat{\mathbf{n}}_1 + u_2\hat{\mathbf{n}}_2 + u_3\hat{\mathbf{n}}_3 \tag{62} $$

where $\hat{\mathbf{n}}_1$, $\hat{\mathbf{n}}_2$, and $\hat{\mathbf{n}}_3$ are a right-handed set of mutually perpendicular unit vectors fixed in N. Appell's restriction on the velocity of P is often expressed by the relationship

$$ {u_3}^2 = a^2({u_1}^2 + {u_2}^2) \tag{63} $$

where a is a constant. It is pointed out by Smith that the relationship describes a requirement for the angle γ between ${}^N\mathbf{v}^P$ and $\hat{\mathbf{n}}_3$, the vertical direction, to remain constant. In fact, the constant a is $\cos\gamma/\sin\gamma$. The nonlinear nonholonomic constraint equation is differentiated with respect to time to bring it to the acceleration level

$$ 2u_3\dot{u}_3 = 2a^2(u_1\dot{u}_1 + u_2\dot{u}_2) \tag{64} $$

where it is linear in $\dot{u}_1$, $\dot{u}_2$, and $\dot{u}_3$; it can be rewritten as

$$ {}^N\mathbf{a}^P \cdot \hat{\mathbf{n}}_3 - \frac{a^2}{u_3}(u_1\,{}^N\mathbf{a}^P \cdot \hat{\mathbf{n}}_1 + u_2\,{}^N\mathbf{a}^P \cdot \hat{\mathbf{n}}_2) = {}^N\mathbf{a}^P \cdot \left[\hat{\mathbf{n}}_3 - \frac{a^2}{u_3}(u_1\hat{\mathbf{n}}_1 + u_2\hat{\mathbf{n}}_2)\right] = 0 \tag{65} $$

where ${}^N\mathbf{a}^P$ is the acceleration of P in N. Inspection of this equation according to Eqs. (2) and (3) indicates that a constraint force $\mathbf{C}$ must be applied to P such that the force is parallel to the vector within the square brackets; that is,

$$ \mathbf{C} = \lambda\left[\hat{\mathbf{n}}_3 - \frac{a^2}{u_3}(u_1\hat{\mathbf{n}}_1 + u_2\hat{\mathbf{n}}_2)\right] \tag{66} $$

This result is in agreement with what is presented by Smith, who shows that $\mathbf{C} \cdot {}^N\mathbf{v}^P = 0$ when ${}^N\mathbf{v}^P$ obeys the constraint. The advantage of inspecting Eq. (65) and immediately obtaining the vector form in Eq. (66) is readily apparent; the result is hardly obvious.

The gravitational force acting on P is denoted by $\mathbf{f} = -mg\hat{\mathbf{n}}_3$ where m is the mass of P and the constant g represents the gravitational force per unit mass. Three dynamical equations of motion obtained with Eqs. (4) can be written in terms of vectors as $\hat{\mathbf{n}}_r \cdot (\mathbf{f} + \mathbf{C} - m\,{}^N\mathbf{a}^P) = 0$ $(r = 1, 2, 3)$, or in terms of scalars

$$ m\dot{u}_1 = -\lambda a^2 u_1/u_3, \qquad m\dot{u}_2 = -\lambda a^2 u_2/u_3, \qquad m\dot{u}_3 = \lambda - mg \tag{67} $$

in which case they resemble certain expressions found by Smith. When one substitutes u_3 obtained from the constraint equation (63), the results are identical to Eqs. (3.7) of Ref. [23],

$$m\dot{u}_1 = -\lambda a \frac{u_1}{\sqrt{{u_1}^2 + {u_2}^2}}, \qquad m\dot{u}_2 = -\lambda a \frac{u_2}{\sqrt{{u_1}^2 + {u_2}^2}}, \qquad m\dot{u}_3 = \lambda - mg \tag{68}$$

The fourth relationship needed to determine the unknowns $\dot{u}_1$, $\dot{u}_2$, $\dot{u}_3$, and λ is provided by Eq. (64); when it is solved for $\dot{u}_3$ and substitution is performed in the third of Eqs. (68), one obtains

$$\lambda = mg + m\frac{a^2}{u_3}(u_1\dot{u}_1 + u_2\dot{u}_2) = mg - \frac{a}{\sqrt{{u_1}^2 + {u_2}^2}}\left[\frac{\lambda a({u_1}^2 + {u_2}^2)}{\sqrt{{u_1}^2 + {u_2}^2}}\right] = mg - \lambda a^2 \tag{69}$$

where the second step is made with the aid of Eq. (63) together with the first and second of Eqs. (68). A solution for λ is now at hand, and it can be used as a replacement in the first and second of Eqs. (68) to yield

$$\lambda = \frac{mg}{1+a^2} = mg\sin^2\gamma \tag{70}$$

$$\dot{u}_1 = -\frac{gau_1}{(1+a^2)\sqrt{{u_1}^2 + {u_2}^2}} = -\frac{g\sin\gamma\cos\gamma\, u_1}{\sqrt{{u_1}^2 + {u_2}^2}} \tag{71}$$

$$\dot{u}_2 = -\frac{gau_2}{(1+a^2)\sqrt{{u_1}^2 + {u_2}^2}} = -\frac{g\sin\gamma\cos\gamma\, u_2}{\sqrt{{u_1}^2 + {u_2}^2}} \tag{72}$$

The dynamical equations of motion (71) and (72), which do not contain λ, can be obtained directly instead by resorting to Eqs. (5). After embedding the acceleration level constraint equation in ${}^N\mathbf{a}^P$,

$${}^N\mathbf{a}^P = \dot{u}_1\hat{\mathbf{n}}_1 + \dot{u}_2\hat{\mathbf{n}}_2 + \frac{a(u_1\dot{u}_1 + u_2\dot{u}_2)}{\sqrt{{u_1}^2 + {u_2}^2}}\hat{\mathbf{n}}_3 \tag{73}$$

the required nonholonomic partial accelerations of P in N are readily identified to be

$${}^N\tilde{\mathbf{a}}_1^P = \hat{\mathbf{n}}_1 + \frac{au_1}{\sqrt{{u_1}^2 + {u_2}^2}}\hat{\mathbf{n}}_3, \qquad {}^N\tilde{\mathbf{a}}_2^P = \hat{\mathbf{n}}_2 + \frac{au_2}{\sqrt{{u_1}^2 + {u_2}^2}}\hat{\mathbf{n}}_3 \tag{74}$$

The two equations of interest are then produced from ${}^N\tilde{\mathbf{a}}_r^P \cdot (\mathbf{f} + \mathbf{C} - m\,{}^N\mathbf{a}^P) = {}^N\tilde{\mathbf{a}}_r^P \cdot (\mathbf{f} - m\,{}^N\mathbf{a}^P) = 0$ $(r = 1, 2)$. Although some effort is required because the equations are coupled in $\dot{u}_1$ and $\dot{u}_2$, Eqs. (71) and (72) are recovered. No Jacobian or orthogonal complement matrices are involved in obtaining the results in this fashion.

6 A System Containing a Rigid Body

Huston and Passerello (Ref. [25]) were the first to approach the matter of extending Kane's method to deal with nonlinear nonholonomic constraint equations; their work is refined in Ref. [11]. A similar viewpoint for dealing with linear nonholonomic constraint equations is presented in Refs. [31] and [10].

There are certain concepts that the exposition in Sec. 2 has in common with that of Ref. [11]. The authors of that work recognize constraint equations that are nonlinear at the velocity level become linear at the acceleration level, and they note the relationship between partial acceleration and partial velocity expressed in Eqs. (9). They make use of these observations to form equations of motion that are equivalent to Eqs. (4), and form generalized constraint forces that are expressed with the final term in Eqs. (19). It is pointed out that the undetermined multipliers can be eliminated and a reduced set of equations of motion can be obtained.

There exist a number of differences between what is presented here and in Ref. [11]. In that work, the development is restricted to motion variables that are each defined as the time derivative of a single generalized coordinate. Remainder terms such as ${}^{N}\mathbf{v}_t^{P_i}$ or ${}^{N}\tilde{\mathbf{v}}_t^{P_i}$ needed to account for prescribed motion are not included in the formulation. The development requires partial velocities to be expressed in a vector basis fixed in an inertial reference frame, which is not necessarily convenient or efficient. In contrast, the motion variables used here are fully general linear combinations as in Eqs. (2.12.1) of Ref. [28], velocity remainder terms ${}^{N}\tilde{\mathbf{v}}_t^{P_i}$ are included [see Eqs. (10)], and all partial velocities (for that matter, all vectors) introduced herein are considered basis-independent quantities just as they are in Ref. [28].

In Ref. [11], equations containing the multipliers are formed first; the multipliers are subsequently eliminated and a reduced set of equations of motion similar to Eqs. (5) is obtained by premultiplication with an orthogonal complement matrix. (An analogous approach is taken in Refs. [10] and [31] in connection with linear nonholonomic constraint equations.) As is well known, an orthogonal complement is not unique. In simple problems an orthogonal complement can be obtained analytically, as in Ref. [11]. Usually, however, it is produced numerically via the zero-eigenvalue theorem, singular value decomposition, QR decomposition, successive multiplication of Householder transformations, etc. As noted earlier, the Appell-Hamel mechanism is used to illustrate the method proposed in Ref. [11] even though it involves contrived nonlinearity in nonholonomic constraint equations.

The present work puts forth two significant advances over the material in Ref. [11]. First, information about the direction and point of application of constraint forces is obtained by inspecting constraint equations written in vector form at the acceleration level. As demonstrated later in this section, the direction and body of application of a constraint torque can be obtained in the same way. In Ref. [11] the undetermined multipliers are related in a clear way to scalar generalized constraint forces, but not to constraint forces and torques in vector form. Second, it is discovered here that *nonholonomic* partial accelerations can be used to construct Eqs. (5) directly and analytically. This approach circumvents the need to form Eqs. (4) first and, afterwards, carry out what are usually two numerical procedures, namely production and application of an orthogonal complement. The absence of orthogonal complements is a desirable feature common to the methods of Ref. [28] and this work. There is no introduction of the nonholonomic partial acceleration in Ref. [11], or of the nonholonomic partial angular acceleration that is defined in what follows. In contrast to nonunique orthogonal complements, the nonholonomic partial accelerations and nonholonomic partial angular accelerations proposed here are unique once a set of independent motion variable time derivatives has been chosen, and they are formed by the same definite process of inspection used to obtain partial velocities and nonholonomic partial velocities.

The apparatus of Ref. [11] deals with the practical case of a system containing rigid bodies rather than the general case of a system containing individual particles. What follows is a presentation of the essential steps needed to extend the discussion in Sec. 2 to encompass rigid bodies. The results allow one to deal with an inherently nonlinear nonholonomic constraint equation such as ${}^N\boldsymbol{\omega}^A \cdot {}^N\boldsymbol{\omega}^B = 0$, where ${}^N\boldsymbol{\omega}^A$ and ${}^N\boldsymbol{\omega}^B$ are the angular velocities in an inertial reference frame N of two unconnected rigid bodies A and B respectively. One is then in a position to identify the directions of the constraint torques that must be applied to

A and B in order to keep ${}^N\boldsymbol{\omega}^A$ perpendicular to ${}^N\boldsymbol{\omega}^B$. It also becomes possible to derive, directly, explicit analytical equations that govern the constrained motion of the two bodies even though the equations are devoid of multipliers.

When particles $P_1, \ldots, P_\beta$ make up a rigid body B, the acceleration ${}^N\mathbf{a}^{P_i}$ in N of a generic particle P_i of B can be written in terms of the angular acceleration ${}^N\boldsymbol{\alpha}^B$ of B in N, the angular velocity ${}^N\boldsymbol{\omega}^B$ of B in N, and the acceleration ${}^N\mathbf{a}^{B^\star}$ in N of $B^\star$, the mass center of B,

$$ {}^N\mathbf{a}^{P_i} = {}^N\mathbf{a}^{B^\star} + {}^N\boldsymbol{\alpha}^B \times \mathbf{r}_i + {}^N\boldsymbol{\omega}^B \times ({}^N\boldsymbol{\omega}^B \times \mathbf{r}_i) \qquad (i = 1, \ldots, \beta) \tag{75} $$

where $\mathbf{r}_i$ is the position vector from $B^\star$ to P_i. Now, ${}^N\boldsymbol{\alpha}^B$ can be expressed uniquely as

$$ {}^N\boldsymbol{\alpha}^B = \sum_{r=1}^{c} {}^N\tilde{\boldsymbol{\alpha}}_r^B\, \dot{u}_r + {}^N\tilde{\boldsymbol{\alpha}}_t^B \tag{76} $$

where ${}^N\tilde{\boldsymbol{\alpha}}_r^B$ is called the rth *nonholonomic partial angular acceleration* of B in N. Substitution from this relationship and from Eqs. (8) into (75) yields

$$ \begin{aligned} \sum_{r=1}^{c} {}^N\tilde{\mathbf{a}}_r^{P_i}\, \dot{u}_r + {}^N\tilde{\mathbf{a}}_t^{P_i} &= \sum_{r=1}^{c} {}^N\tilde{\mathbf{a}}_r^{B^\star}\, \dot{u}_r + {}^N\tilde{\mathbf{a}}_t^{B^\star} \\ &\quad + \left(\sum_{r=1}^{c} {}^N\tilde{\boldsymbol{\alpha}}_r^B\, \dot{u}_r + {}^N\tilde{\boldsymbol{\alpha}}_t^B \right) \times \mathbf{r}_i + {}^N\boldsymbol{\omega}^B \times ({}^N\boldsymbol{\omega}^B \times \mathbf{r}_i) \quad (i = 1, \ldots, \beta) \end{aligned} \tag{77} $$

from which one obtains

$$ {}^N\tilde{\mathbf{a}}_t^{P_i} = {}^N\tilde{\mathbf{a}}_t^{B^\star} + {}^N\tilde{\boldsymbol{\alpha}}_t^B \times \mathbf{r}_i + {}^N\boldsymbol{\omega}^B \times ({}^N\boldsymbol{\omega}^B \times \mathbf{r}_i) \qquad (i = 1, \ldots, \beta) \tag{78} $$

and

$$ {}^N\tilde{\mathbf{a}}_r^{P_i} = {}^N\tilde{\mathbf{a}}_r^{B^\star} + {}^N\tilde{\boldsymbol{\alpha}}_r^B \times \mathbf{r}_i \qquad (r = 1, \ldots, c;\ i = 1, \ldots, \beta) \tag{79} $$

The latter relationship is the nonholonomic partial acceleration analog to nonholonomic partial velocity expressions like Eqs. (4.6.5) and (4.11.16) in Ref. [28] used in the case of simple nonholonomic systems to obtain contributions of B to $\tilde{F}_r$ and $\tilde{F}_r^\star$. Hence, the contribution of B to $\tilde{\tilde{F}}_r$ is given by

$$ \begin{aligned} (\tilde{\tilde{F}}_r)_B &\triangleq \sum_{i=1}^{\beta} {}^N\tilde{\mathbf{a}}_r^{P_i} \cdot \mathbf{R}_i \\ &= \sum_{i=1}^{\beta} \left({}^N\tilde{\mathbf{a}}_r^{B^\star} + {}^N\tilde{\boldsymbol{\alpha}}_r^B \times \mathbf{r}_i \right) \cdot \mathbf{R}_i = {}^N\tilde{\mathbf{a}}_r^{B^\star} \cdot \sum_{i=1}^{\beta} \mathbf{R}_i + {}^N\tilde{\boldsymbol{\alpha}}_r^B \cdot \sum_{i=1}^{\beta} \mathbf{r}_i \times \mathbf{R}_i \\ &= {}^N\tilde{\mathbf{a}}_r^{B^\star} \cdot \mathbf{R} + {}^N\tilde{\boldsymbol{\alpha}}_r^B \cdot \mathbf{T} \qquad (r = 1, \ldots, c) \end{aligned} \tag{80} $$

where the set of all contact and distance forces $\mathbf{R}_i$ acting on the particles of B is equivalent to a force $\mathbf{R}$ whose line of action passes through $B^\star$, together with a couple whose torque is $\mathbf{T}$. The constraint forces and torques that must be applied to B in order to satisfy nonlinear nonholonomic constraint equations may be included in $\mathbf{R}$ and $\mathbf{T}$, or they may be omitted; in either case they will not contribute in aggregate to $\approx{F}_r$. With a similar exercise the contribution of B to $\approx{F}_r^\star$ is found to be

$$
\begin{aligned}
(\approx{F}_r^\star)_B &\stackrel{\triangle}{=} -\sum_{i=1}^{\beta} {}^N\tilde{\mathbf{a}}_r^{P_i} \cdot m_i \, {}^N\mathbf{a}^{P_i} \\
&= -\sum_{i=1}^{\beta} \left({}^N\tilde{\mathbf{a}}_r^{B^\star} + {}^N\tilde{\boldsymbol{\alpha}}_r^B \times \mathbf{r}_i \right) \cdot m_i \, {}^N\mathbf{a}^{P_i} \\
&= -\, {}^N\tilde{\mathbf{a}}_r^{B^\star} \cdot \sum_{i=1}^{\beta} m_i \, {}^N\mathbf{a}^{P_i} - {}^N\tilde{\boldsymbol{\alpha}}_r^B \cdot \sum_{i=1}^{\beta} \mathbf{r}_i \times m_i \, {}^N\mathbf{a}^{P_i} \\
&= {}^N\tilde{\mathbf{a}}_r^{B^\star} \cdot \mathbf{R}^\star + {}^N\tilde{\boldsymbol{\alpha}}_r^B \cdot \mathbf{T}^\star \qquad (r = 1, \dots, c)
\end{aligned}
\tag{81}
$$

where $\mathbf{R}^\star$ and $\mathbf{T}^\star$ are, respectively, the well-known inertia force and inertia torque for B in N, formed for use with Kane's method.

The procedure for obtaining directly a minimal set of dynamical equations of motion for a complex nonholonomic system is seen to bear a very close resemblance to Kane's method for simple nonholonomic systems, the only difference being that one uses ${}^N\tilde{\mathbf{a}}_r^{B^\star}$ and ${}^N\tilde{\boldsymbol{\alpha}}_r^B$ $(r = 1, \dots, c)$, vectors that are distinct from the familiar vectors ${}^N\tilde{\mathbf{v}}_r^{B^\star}$ and ${}^N\tilde{\boldsymbol{\omega}}_r^B$ $(r = 1, \dots, p)$.

One may be interested in the constraint forces acting on a rigid body, and therefore form equations of motion according to Eqs. (4). In that event it becomes desirable to adapt the process of inspecting a constraint equation written at the acceleration level so that one may identify the direction of a constraint force and the point to which it is applied, together with the direction of a constraint torque and the body upon which it is exerted.

In a constraint equation having the form of (2), the terms associated with $P_1, \dots, P_\beta$ can be rewritten:

$$
\begin{aligned}
&\sum_{i=1}^{\beta} {}^N\mathbf{a}^{P_i} \cdot \mathbf{W}_{is} + Z_s \\
&\qquad = \sum_{i=1}^{\beta} [\, {}^N\mathbf{a}^{Q} + {}^N\boldsymbol{\alpha}^B \times \mathbf{r}_i + {}^N\boldsymbol{\omega}^B \times ({}^N\boldsymbol{\omega}^B \times \mathbf{r}_i)] \cdot \mathbf{W}_{is} + Z_s
\end{aligned}
$$

$$= {}^N\mathbf{a}^Q \cdot \sum_{i=1}^{\beta} \mathbf{W}_{is} + {}^N\boldsymbol{\alpha}^B \cdot \sum_{i=1}^{\beta} \mathbf{r}_i \times \mathbf{W}_{is} + \sum_{i=1}^{\beta} [{}^N\boldsymbol{\omega}^B \times ({}^N\boldsymbol{\omega}^B \times \mathbf{r}_i)] \cdot \mathbf{W}_{is} + Z_s$$

$$\triangleq {}^N\mathbf{a}^Q \cdot \mathbf{W}_s + {}^N\boldsymbol{\alpha}^B \cdot \boldsymbol{\tau}_s + Z'_s \qquad (s = 1, \ldots, l) \tag{82}$$

where $\mathbf{r}_i$ is the position vector from a point Q fixed in B to P_i ($i = 1, \ldots, \beta$). The point Q need not be the mass center of B. As discussed in connection with Eqs. (2) and (3), the appearance of the vector $\mathbf{W}_{is}$ in Eqs. (82) requires the application of a constraint force $\mathbf{C}_{is} = \lambda_s \mathbf{W}_{is}$ to P_i. After selecting the line of action of $\mathbf{W}_{is}$ such that it passes through P_i, and defining the resultants

$$\mathbf{W}_s \triangleq \sum_{i=1}^{\beta} \mathbf{W}_{is}, \qquad \mathbf{C}_s \triangleq \sum_{i=1}^{\beta} \mathbf{C}_{is} \qquad (s = 1, \ldots, \ell) \tag{83}$$

the set of forces $\mathbf{C}_{1s}, \ldots, \mathbf{C}_{\beta s}$ applied to B is regarded as equivalent to a single force $\mathbf{C}_s$ whose line of action passes through Q, together with a couple whose torque is $\mathbf{T}_s$. The resultant $\mathbf{C}_s$ is given by

$$\mathbf{C}_s = \sum_{i=1}^{\beta} \mathbf{C}_{is} = \sum_{i=1}^{\beta} \lambda_s \mathbf{W}_{is} = \lambda_s \mathbf{W}_s \qquad (s = 1, \ldots, \ell) \tag{84}$$

and the torque $\mathbf{T}_s$ is equal to the moment of $\mathbf{C}_{1s}, \ldots, \mathbf{C}_{\beta s}$ about Q,

$$\mathbf{T}_s = \sum_{i=1}^{\beta} \mathbf{r}_i \times \mathbf{C}_{is} = \sum_{i=1}^{\beta} \mathbf{r}_i \times \lambda_s \mathbf{W}_{is} = \lambda_s \boldsymbol{\tau}_s \qquad (s = 1, \ldots, \ell) \tag{85}$$

where $\boldsymbol{\tau}_s$ is the moment of $\mathbf{W}_{1s}, \ldots, \mathbf{W}_{\beta s}$ about Q,

$$\boldsymbol{\tau}_s \triangleq \sum_{i=1}^{\beta} \mathbf{r}_i \times \mathbf{W}_{is} \qquad (s = 1, \ldots, \ell) \tag{86}$$

One can therefore inspect a constraint equation written at the acceleration level and conclude that the appearance of the dot product ${}^N\mathbf{a}^Q \cdot \mathbf{W}_s$ requires that B is subject to a constraint force $\mathbf{C}_s = \lambda_s \mathbf{W}_s$ applied to Q, and the appearance of the dot product ${}^N\boldsymbol{\alpha}^B \cdot \boldsymbol{\tau}_s$ means B must be acted upon by a couple whose constraint torque is $\mathbf{T}_s = \lambda_s \boldsymbol{\tau}_s$ ($s = 1, \ldots, \ell$).

The contribution of B to Eqs. (4) is thus represented by

$$(\tilde{F}_r^\star)_B = {}^N\tilde{\mathbf{v}}_r^{B^\star} \cdot \mathbf{R}^\star + {}^N\tilde{\boldsymbol{\omega}}_r^B \cdot \mathbf{T}^\star, \qquad (\tilde{F}_r)_B = {}^N\tilde{\mathbf{v}}_r^Q \cdot \mathbf{R} + {}^N\tilde{\boldsymbol{\omega}}_r^B \cdot \mathbf{T} \qquad (r = 1, \ldots, p) \tag{87}$$

where the set of all contact forces and distance forces $\mathbf{R}_i$ acting on the particles of B is equivalent to a force $\mathbf{R}$ whose line of action passes through Q, together with a couple whose torque is $\mathbf{T}$. All constraint forces $\mathbf{C}_s$ applied to Q are included in the resultant $\mathbf{R}$, and all constraint torques $\mathbf{T}_s$ exerted on B are included in $\mathbf{T}$. The vectors ${}^N\tilde{\mathbf{v}}_r^Q$ and ${}^N\tilde{\boldsymbol{\omega}}_r^B$ are (Ref. [28]), respectively, the rth nonholonomic partial velocity of Q in N and the rth nonholonomic partial angular velocity of B in N. If the system S to which B belongs is not subject to motion constraints described by equations that are inherently nonlinear in velocity ($\ell = 0$), then S is a simple nonholonomic system and Eqs. (87) become precisely the relationships provided in Ref. [28] for such a system. If all nonlinearities in the nonholonomic constraint equations are contrived, then S is in fact a simple nonholonomic system and should be treated as such.

7 Conclusions

In dealing with motion constraints that are expressed at the velocity level with relationships that are nonlinear in velocity, there is a distinction to be made between nonholonomic constraint equations in which the nonlinearity is inherent, and those in which the nonlinearity is contrived. Methods are proposed in this paper for dealing with equations of the former type.

Certain forces and torques are required to ensure satisfaction of nonholonomic constraint equations that are inherently nonlinear in velocity. One may be interested in expressing these constraint forces and torques in vector form so that their directions are known, and there may also be interest in knowing the specific points at which the constraint forces must be applied or the particular bodies upon which the constraint torques are to be exerted. In that case, one may write constraint equations at the acceleration level in vector form, in terms of dot products of vectors, and determine the desired information by the simple process of inspection. Such information is not available from any of the methods found in the existing literature, where constraint equations are invariably expressed in scalar or matrix form. The methodology presented herein provides the information readily and stands as one of the paper's main contributions. As demonstrated here by several examples, this method is especially advantageous in cases where the required direction of a constraint force is not otherwise obvious.

When one wishes to construct equations containing evidence of constraint forces and constraint torques, solution of which yields time histories of those forces and torques, one forms equations of motion with Kane's method as though a simple nonholonomic system is involved. The vector expressions for constraint forces and torques obtained by inspection are included with the vector expressions for the usual applied forces and torques. In this way, generalized constraint forces are obtained by using a fundamental definition involving dot products of vectors, rather than by forming the product of a Jacobian matrix and an array of multipliers as recommended in the current literature. The multipliers introduced here bear a clear relationship to constraint force and torque vectors, whereas this is not the case with other methods.

On the other hand, when the constraint forces and torques in question are not of interest, one may form equations of motion that do not involve those forces and torques in any way. Such equations can be constructed directly, in explicit analytical form, without first formulating equations that do contain evidence of the constraint forces and torques. This is accomplished by employing vectors known as nonholonomic partial accelerations and nonholonomic partial angular accelerations; these vectors are distinct from the well-known nonholonomic partial velocities and nonholonomic partial angular velocities used to form Kane's equations for simple nonholonomic systems, and they are obtained with the same simple process of inspection. The use of an orthogonal complement matrix is required when one employs existing extensions made to Kane's method for the purpose of dealing with nonlinear nonholonomic constraint equations. Construction of minimal equations of motion without resorting to an orthogonal complement represents a significant advantage over such approaches, and constitutes another major contribution of the paper.

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